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Adapted Fourier transform of Schwartz spaces for certain nilpotent Lie groups

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Abstract. *In the framework of the deformation program* ($*$*-product*), *initiated in [1], a nilpotent Fourier transform* Θ *was introducted ([4], [5]). This transformation is defined for* f *in* $S(G)$: *a dense invariant open subset* F *of* g^* *is parametrized by* $F = \mathbf{R}^{2k} \times \mathcal{V}$ *where* $\mathcal{V}$ *is the set of coadjoint orbits in* F *and:*

$$\Theta(f)(p,q,\lambda) = \int_{\mathbf{g}} \exp(i a(X,p,q,\lambda)) f(\exp X) \, \mathrm{d}X \text{ where}$$

$$X \in \mathbf{g}, (p,q,\lambda) \in \mathbf{R}^{2k} \times \mathcal{V},$$

a is a rational function in λ, *polynomial in* (X,p,q). *If* G *is abelian,* Θ *is the usual Fourier transform.*

In [6] we have proved that $\Theta : S(G) \to C^\infty(V, S(p,q))$ *is continuous where* $C^\infty(\mathcal{V}, S(p,q))$ *is the space of* C^∞ *functions in* $\lambda \in \mathcal{V}$ *to* $S(pq)(S(p,q)$ *is the schwartz space on* $\mathbf{R}^{2k}$).

In this paper, we give a criterium for the density of $\Theta(S(G))$ *in* $C^\infty(V, S(p,q))$ *(hypothesis* (H) *of section 2) and many examples where this hypothesis holds.*

INTRODUCTION

The implementation of the deformation theory in the harmonic analysis is more explicitly translatable for the nilpotent groups; in this case we define a $*$-product (i. e. a deformation of usual product of C^∞ functions on F, where F is an invariant (for the coadjoint representation) Zariski open set of $\mathbf{g}^*$), this is possible thanks to the fact that

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we are able to simultaneously and globaly parametrize the orbits of F and then to define on each one the Moyal $*$-product (see section 1).

Moreover a nilpotent Fourier transform was introduced between $L^2(G)$ and $L^2(F)$ ([4], [5]) and this is a generalisation of the usual Fourier transform for the abelian case; Θ is a unitary transformation and has numerous properties. In our opinion Θ has to play an important role as the usual Fourier transform for the abelian case.

This paper is a contribution to the study of this transformation, we will be interested here in the question of the range by Θ of functional spaces on G. Between these spaces, one of the most important is: $S(G)$ (the space of functions f on G such that $f \circ \exp$ is a C^∞ function, rapidly decreasing on $\mathbf{g}$), on the other hand, the parametrization of F proves that F is diffeomorphic to $\mathcal{V} \times \mathbf{R}^{2k}$, where $\mathcal{V}$ is a Zariski open set of $\mathbf{R}^{n-2k}$ ($\dim \mathbf{g} = n$). $\mathcal{V}$ is also the G-orbits space of F.

We know that [5] $\Theta(S(G))$ is included in $C^\infty(\mathcal{V}, S(\mathbf{R}^{2k}))$, in [6] we proved that Θ is a continuous mapping from $S(G)$ to $C^\infty(\mathcal{V}, S(\mathbf{R}^{2k}))$ if we endow these spaces with their usual topologies.

Here we shall be interested by the question of the density of $\Theta(S(G))$ in $C^\infty(\mathcal{V}, S(\mathbf{R}^{2k}))$, more precisely we shall prove the density under a hypothesis (H) on the parametrization of F (see section 2), and finally we give many examples of nilpotent Lie groups for which the hypothesis (H) holds in the last section.

SECTION 1.

We denote by G a nilpotent connected and simply connected Lie group, $\mathbf{g}$ the Lie algebra of G with dual g^*, and $U(\mathbf{g})$, $S(\mathbf{g})$, $I(\mathbf{g})$ are respectively the universal enveloping algebra, the symetric algebra of $\mathbf{g}$ and the ring of the G-invariant polynomial functions on $\mathbf{g}^*$

The unitary dual $\hat{G}$ of G was determined by Kirillov [3]: there exists a one to one mapping between the set of co-adjoint orbits in $\mathbf{g}^*$ and the set of equivalence classes of unitary irreductible representations of G, this correspondance betweeen the orbit O^λ and the representation π^λ is explicitely determined by the character formula of Kirillov:

$$Tr(\pi^\lambda(f)) = \int_{O^\lambda} \hat{f}(\xi) d\mu_\lambda(\xi) \qquad \text{for } f \in S(G)$$

Where $S(G)$ is the space of the functions f with $f \circ \exp$ is in the Schwartz space of $\mathbf{g}$, $\hat{f}$ is the usual Fourier transform of $f \circ \exp$ and $d\mu_\lambda$ is the canonical measure on O^λ; $\pi^\lambda(f)$ is a Hilbert Scmidt operator on $H = L^2(\mathbf{R}^k)$($k = 1/2 \dim O^\lambda$, H is the space of realization of π^λ).

First we choose a parametrization of a dense invariant subset F of $\mathbf{g}^*$ the result is as follows:

THEOREM 1. [2] *There exists a Zariski open set* $F \subset \mathbf{g}^*$, *rational functions* $p_j, q_j, (j = 1, \dots, k, k$ *is* $1/2$ *of* $\dim O^\lambda)$ *and rational functions* $\lambda_m (m = 1, \dots, r; r = \dim \mathbf{g} - 2k)$; p_j, q_j, λ_m *being smooth on* F *such that:*

1) *the map* $\Phi : F \to \mathcal{V} \times \mathbb{R}^{2k}$ $(\xi \to (p, (\xi), q(\xi), \lambda(\xi)))$ *is a diffeomorphism here* $\mathcal{V} = \lambda(F)$ *is a Zariski open subset of* $\mathbb{R}^r$

2) *each orbit* O^λ *in* F *is* $O^\lambda = \Phi^{-1}(\{\lambda\} \times \mathbb{R}^{2k})$ *and the map* $O^\lambda \to \mathbb{R}^{2k}$ $(\xi \to (p(\xi), q(\xi)))$ *is a diffeomorphism. In particular* $(2\pi)^k \mathrm{d}\,\mu_\lambda(\xi)$ *and the canonical symplectic structure of* O^λ *is defined by the 2-form* $\sum \mathrm{d}\,p_j \wedge \mathrm{d}\,q_j$.

3) $\forall X \in \mathbf{g}, \tilde{X}(p, q, \lambda) = \langle X, \xi \rangle = \sum_{j=1}^{k} \alpha_j(\lambda, q) p_j + \alpha_0(\lambda, q)$ *where* α_j *is in* $\mathbb{R}(\lambda)[q_{j+1}, \dots, q_k]$.

As this parametrization plays a central role in this article, we shall briefly recall how it is obtained. Our parametrization (p, q, λ) is builded up by induction on $n = \dim \mathbf{g}$, following a sequence $\{o\} \subset \mathbf{g}_1 \subset \dots \subset \mathbf{g}_n = \mathbf{g}$, where $\mathbf{g}_i$ is an i-dimensional ideal of $\mathbf{g}$. Let X_i be an element of $\mathbf{g}_i - \mathbf{g}_{i-1}$, thus $\{X_i, i = 1, \dots, n\}$ is a basis of $\mathbf{g}$. As in [2] there are two possibilities:

CASE A. $I(g_n) \not\subset S(g_{n-1})$ then $I(g_{n-1}) \subset I(g_n)$ and there exists λ in $I(g_n) - I(g_{n-1})$ such that:

$$\lambda_r = \alpha \tilde{X}_n + \beta \qquad (\alpha \in I(\mathbf{g}_{n-1}), \beta \in S(g_{n-1}))$$

and $(p, q) = (p', q'); \lambda = (\lambda', \lambda_r)$.

CASE B. $I(\mathbf{g}_n) \subset S(\mathbf{g}_{n-1})$ then $I(\mathbf{g}_n) \subset I(\mathbf{g}_{n-1})$ and there exists Y in $I(\mathbf{g}_{n-1}) - I(\mathbf{g}_n)$ such that:

$$\{\tilde{X}_n, \tilde{Y}\} = \tilde{Z}, \quad \tilde{Z} \in I(\mathbf{g}_n) \qquad \text{and} \qquad \tilde{Z} \neq 0$$

$\{,\}$ is the Poisson bracket on O^λ. We put:

$$p_k = \tilde{X}_n, \quad q_k = \tilde{Y}.\tilde{Z}^{-1}, \quad p_i = \exp -q_k \mathrm{ad}_{S(\mathbf{g})} \tilde{X}_n p_i',$$
$$q_i = \exp -q_k \mathrm{ad}_{S(\mathbf{g})} \tilde{X}_n q_i'.$$

Now we can write the Plancherel formula for G in the following form:

$$f(e) = \int_{\mathbf{g}^*} \hat{f}(\xi) \mathrm{d}\,\xi = \int_{\mathcal{V}} Tr(\pi^\lambda(f)) r(\lambda) \mathrm{d}\,\lambda$$

where $\mathrm{d}\,x = r(\lambda) \mathrm{d}\,\lambda \mathrm{d}\,p \mathrm{d}\,q, r(\lambda)$ is rational, regular on $\mathcal{V}$.

The nilpotent Fourier transform is introduced by a deformation of Poisson bracket and associative product of C^∞ functions on $\mathbf{g}^*$ [4].

Precisely, we define on $S(O^\lambda) = S(p,q)$ the $*$ product:

$$u_* v(\xi) = \pi^{-k} \int_{O^\lambda \times O^\lambda} u(\xi') v(\xi'') e^{2i[\beta(\xi,\xi'')+\beta(\xi'',\xi')+\beta(\xi',\xi)]} d\xi' d\xi''.$$

where $\quad \beta(\xi,\xi') = \sum_{j=1}^{k} p_j q'_j - p'_j q_j$

and $\quad \xi = (p,q), \xi' = (p',q')$.

This $*$ product can be extented to $L^2(O^\lambda)$, [1]. On the other hand, if u or v is a polynomial function, we put:

$$u_* v(\xi) = \sum_{m=0}^{\infty} \left(\frac{1}{2i}\right)^m \frac{1}{m!} \Lambda^{i_1 j_1} \dots \Lambda^{i_r j_r} \partial_{i_1 \dots i_r} u \partial_{j_1 \dots j_r} v(\xi)$$

Where Λ^{ab} is the $2k \times 2k$ matrix $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ and $\partial_{i_1 \dots i_r} = \dfrac{\partial^r}{\partial X_{i_1} \dots \partial X_{i_r}}$ with $\{X_1, \dots, X_{2k}\} = \{p_1, \dots, p_k, q_1, \dots, q_k\}$.

$(S(O^\lambda), *)$ is an associative and involutive for complex conjugation algebra. If we put, for u, v in $S(O^\lambda)$:

$$\ell(u)v = u_* v, r(u)v = v_* u, \Omega = 2^k e^{-(p^2+q^2)}, \Omega_s =$$

$$\mathop{*}_{j=1}^{k} (p_j + iq_j)^{*s_j} * \Omega 2^{-\frac{|s|}{2}}, s = (s_1, \dots, s_k) \in \mathbb{N}^k$$

we have:

THEOREM 2. [5]

1) $\Omega \in S(O^\lambda), ||\Omega||, \Omega * \Omega = \Omega$

2) *The system* $\{\Omega_s, s$ *is in* $\mathbb{N}^k\}$ *is an orthonormal basis of* $r(\Omega)(L^2(O^\lambda))$

3) *The system* $\{\Omega_s * \bar{\Omega}_m s$, *and* m *being in* $\mathbb{N}^k\}$ *is a hilbert basis of* $L^2(o^\lambda)$

4) $S(O^\lambda)$ *admits one and only one faithful irreductible representation* $T^\lambda(u).\varphi = U_0 l(u)_0 U^{-1}.\varphi$ *with* $U : r(\Omega)L^2(O^\lambda) \to L^2(\mathbf{R}^k)$ *and* $U(\Omega_s) = h_s . h_s = \dfrac{(-i)^{|s|}}{s! 2^{|s|} \sqrt{\pi}} e^{\frac{x^2}{2}} \dfrac{\partial^{|s|} e^{-x^2}}{(\partial x)^s}$.

5) T *is a unitary transformation of* $L^2(O^\lambda)$ *into the space* HS *of Hilbert-Shmidt operators on* H *and:*

$$TrT^\lambda(u) = \int_{O^\lambda} u(p,q,\lambda) \frac{dp.dq}{(2\pi)^k} \qquad (u \in S(O^\lambda))$$

From this point we define the nilpotent Fourier transform of f in $S(G)$ by:

$$\Theta(f)\,\big|_{O^\lambda} = (T^\lambda)^{-1} \circ \pi^\lambda(f)$$

Then it can be shown that $\Theta(f)\,\big|_{O^\lambda}$ is in $S(O^\lambda)$ for each λ in $\mathcal{V}$ and $\Theta(f)$ is a C^∞ function on F. See [4] and [5].

Let $\times$ be the convolution on $S(G)$, then $\Theta(f \times h)$ coincides with $\Theta(f) * \Theta(h)$. Moreover we have an integral formula for this Fourier transform:

THEOREM 3. [5]

1) *There exists a real function* $a(X,\xi)$ *which is polynomial in* X, *rational in* ξ *in* $\mathfrak{g}^*$ *and regular on* F *such that:*

$$\Theta(f)(p,q,\lambda) = \int_{\mathfrak{g}} e^{ia(X,p,q,\lambda)} f(\exp X)\,\mathrm{d}X$$

and

$$f(\exp X) = \int_{\mathfrak{g}^*} \Theta(f) e^{-ia(X,p,q,\lambda)} r(\lambda)\,\mathrm{d}\lambda\,\mathrm{d}p\,\mathrm{d}q.$$

In view of the construction of $\Theta(f)$ ([4], [5]) we can give a more explicit expression of $a(X,p,q,\lambda)$ that is:

$$a(X,p,q,\lambda) = \int_0^1 \tilde{X}(p,\psi,(\exp t\bar{X}(0,\varphi^{-1}(q),\lambda)),\lambda)\,\mathrm{d}t.$$

where:

$$\tilde{X}(p,q,\lambda) = \sum_{j=1}^{k} \alpha_j(X,q,\lambda)p_j + \alpha_0(X,q,\lambda),\ \psi(p,q) = q - \frac{p}{2},$$

$\bar{X}$ is the vector field on O^λ defined by: $\bar{X} = \sum_{j=1}^k \alpha_j(X,\psi,(p,q)) \left(-\frac{\partial}{\partial p_j} + \frac{1}{2}\frac{\partial}{\partial q_j}\right)$

The flow of $\bar{X}$ is $\exp t\bar{X}(p,q) = (P_i^t, Q_i^t)$, where:

$$P_k^t = p_k - t\alpha_k, Q_k^t = q_k + \frac{t}{2}\alpha_k \text{ and } P_1^t = p_i - \int_0^t \alpha_i(\psi(\exp s\bar{X}(p,q,\lambda)))\,\mathrm{d}s$$

$$Q_i^t = q_i - \frac{1}{2}\int_0^t \alpha_i(\psi(\exp s\bar{X}(p,q,\lambda)))\,\mathrm{d}s \text{ finally } \varphi(q_i) = Q_i^t(0,q,\lambda).$$

If f belongs to $S(G)$, with our theorem, the function $\Theta(f)(p,q,\lambda)$ is an element of the space $C^\infty(\mathcal{V}, S(p,q))$ of C^∞-functions of the variable λ to $S(p,q)$. Moreover:

LEMMA 1. $\Theta : L^2(G) \to L^2(F)$ *is an isometry.*

Proof. Let f be in $S(G)$. Using the Plancherel formula we have:

$$\|f\|^2_{L^2(G)} = \int_G \|\pi^\lambda(f)\|^2_{HS}\tau(\lambda)\mathrm{d}\lambda =$$

$$\int_{\mathcal{V}} \|T^\lambda(\Theta(f)\,|_{O^\lambda})\|^2_{HS}\tau(\lambda)\mathrm{d}\lambda$$

but T^λ is a unitary transformation [5], then:

$$\|f\|^2_{L^2(G)} = \int_{\mathcal{V}} \|\Theta(f)|_{O^\lambda}\|^2_{L^2(O^\lambda)}\,\tau(\lambda)\mathrm{d}\lambda = \|\Theta(f)\|^2_{L^2(F)}. \quad \text{Q.E.D.}$$

■

Let ρ(resp. ρ') be the representation of $G \times G$ on $L^2(G)$ (resp. $L^2(\mathbf{g}^*)$) defined by:

$$(\rho(g_1, g_2)f)(g) = f(g_1^{-1} \cdot g \cdot g_2) \qquad \text{for } f \in L^2(G)$$

$$\rho'(\exp X_1, \exp X_2)u = \exp *(-iX_1) * u * \exp *(iX_2)$$

for $u \in L^2(\mathbf{g}^*)$ and X_1, X_2 in $\mathbf{g}$.

As Θ intertwines ρ and ρ' then $Q : L^2(G)^\infty \to L^2(\mathbf{g}^*)^\infty$ is an isomorphism where $L^2(G)^\infty$ (resp. $L^2(\mathbf{g}^*)^\infty$) is the vector space of C^∞ vectors of ρ (resp. ρ').

Now we cut down the more manageable space $S(G)$ wich is dense in $L^2(G)^\infty$ ([10]) hence $\Theta(S(G))$ is dense in $L^2(\mathbf{g}^*)$.

However if f is in $S(G)$ then $\Theta(f)$ is in $C^\infty(\mathcal{V}, S(p,q))$ [5], and:

THEOREM 4. [6] *If $S(G)$ and $C^\infty(\mathcal{V}, S(p,q))$ are endowing with their topology then: $\Theta : S(G) \to C^\infty(\mathcal{V}, S(p,q))$ is continuous.*

Now we ask for the density of $\Theta(S(G))$ in this space.

SECTION 2.

We keep the notations of the previous section, we considere a sequence:

$$\{o\} \subset \mathbf{g}_1 \subset \ldots \subset \mathbf{g}_n = \mathbf{g}.$$

where $\mathbf{g}_i$ is an i-dimensional ideal of $\mathbf{g}$, an element X_i of $\mathbf{g}_i - \mathbf{g}_{i-1}$, thus $\{X_i, i = 1, \ldots, n\}$ is a basis of $\mathbf{g}$ and each g of G can be written:

$$g = \exp \sum_{i=1}^{n} x_1 X_i$$

we choose $(x_1, \ldots, x_n)$ for a system of coordinates on G.

In this section we shall prove that $\Theta(S(G))$ is dense in $C^\infty(\mathcal{V}, S(p,q))$ under the following hypothesis (H) for our parametrization:

$$(H) \begin{cases} \text{for any } i = 1, \ldots, n, \text{ there exists a differential operator} \\ \text{on variables } p, q, \lambda : A^i_{\lambda,p,q} \text{ with coefficients rational} \\ \text{in } \lambda \text{ and polynomial in } p, q \text{ such that:} \\ \quad \Theta(x_i \cdot f) = A^i_{\lambda,p,q} \cdot \Theta(f) \qquad \forall f \in S(G). \end{cases}$$

Our approach is to prove that $\Theta(S(G))$ contains $C_c^\infty(\mathcal{V}, S(p,q)$ where $C_c^\infty(\mathcal{V}, S(p,q))$ is the space of C^∞-compactly supported functions with value in $S(p,q)$.

PROPOSITION 1. *Under the hypothesis (H), if A_x is a differential operator with polynomial coefficients on G then there exists a differential operator $A_{\lambda,p,q}$ with polynomial in (p,q) and rational in λ coefficients such that:*

$$\Theta(A_x . f) = A_{\lambda,p,q} . \Theta(f)$$

Proof. Let X^- be the vector field:

$$X^- f(g) = \frac{\mathrm{d}}{\mathrm{d}t} f(\exp -tX.g)\, \Big|_{t=0} \qquad (f \in C^\infty(G))$$

With our notations:

$$X_1^- = \frac{\partial}{\partial x_1} \quad \text{and} \quad X_j^- = \frac{\partial}{\partial x_j} + \sum_{\ell<j} P_\ell(x_n, \ldots, x_{\ell+1}) \frac{\partial}{\partial x_\ell}$$

here P_ℓ are polynomial functions. But since:

$$p_j * u = p_j \cdot u - \frac{i}{2} \frac{\partial u}{\partial q_j}, \; q_j * u = q_j \cdot u + \frac{i}{2} \frac{\partial u}{\partial p_j}$$

and:

$$\tilde{X}_j = \sum_{i=1}^{k} \alpha_i(X_j, q, \lambda) \cdot p_i + \alpha_0(X_j, q, \lambda)$$

then:

$$\Theta(X_j^- \cdot f) = i\tilde{X}_j * \Theta(f) = B^j_{\lambda,p,q}\Theta(f)$$

where $B^j_{\lambda,p,q}$ is a differential operator with polynomial in (p,q), rational in λ coefficients then by induction on j and using the hypothesis (H) :

$$\Theta\left(\frac{\partial}{\partial x_j}f\right) = \Theta(X_j^- f) - \sum_{i<j}\Theta(P_\ell(x_n,\dots,x_{i+1})\frac{\partial}{\partial x_i}f) = B'^j_{\lambda,p,q}\Theta(f).$$

Then our proposition is a consequence of hypothesis (H).

LEMMA 2. *We suppose the hypothesis (H) holds. Let a be a C^∞, compactly supported function on $\mathcal{V}$; let ψ be defined by:*

$$\psi(\exp X) = \int_F e^{-ia(X,p,q,\lambda)}\alpha(\lambda)\Omega_s * \overline{\Omega_m}r(\lambda)\,d\lambda\, d p\, d q$$
$$(X \in \mathbf{g} \text{ and } (s,m) \in \mathbf{N}^{2k})$$

then ψ is an element of $S(G)$.

Proof. We shall prove that $\psi \in L^2(G)$ and $D\psi \in L^2(G)$ for any differential operator with polynomial coefficients D.

1. Let φ be an element of $S(G)$ then:

$$(\psi,\varphi)_{L^2(G)} = \int_G \overline{\varphi(x)}\left(\int_{\mathcal{V}\times\mathbf{R}^{2k}} e^{-ia(X,p,q,\lambda)}\alpha(\lambda)r(\lambda)\Omega_s.\overline{\Omega_m}\,d\lambda\, d p\, d q\right) d x$$
$$= \int_G \overline{\varphi(x)}\left(\int_{\mathcal{V}} \alpha(\lambda)r(\lambda)(h_s,\pi^\lambda(x)h_m)_{L^2(\mathbf{R}^k)}\,d\lambda\right) d x$$
$$\int_{\mathcal{V}} \alpha(\lambda)r(\lambda)\left(\overline{\int_G (\varphi(x)(\pi^\lambda(x)h_m,h_s)_{L^2(\mathbf{R}^k)}\,d x}\right) d\lambda$$
$$= \int_{\mathcal{V}} \alpha(\lambda)r(\lambda)\left(\Theta(\varphi))\,|_{O^\lambda},\Omega_s.\overline{\Omega_m}\right)_{L^2(\mathbf{R}^{2k})} d\lambda$$
$$= \left(\Theta(\varphi),\alpha(\lambda)\Omega_s.\overline{\Omega_m}\right)_{L^2(F)}$$

thus:

$$|(\psi,\varphi)| \le ||\Theta(\varphi)||_{L^2(F)}\cdot||\alpha.\Omega_s\overline{\Omega_m}||_{L^2(F)} \le C_\psi\cdot||\varphi||_{L^2(G)}.$$

On $L^2(G)$, the linear form $\varphi \to (\psi, \varphi)_{L^2(G)}$ is continuous and by the Riesz theorem, ψ belongs to $L^2(G)$.

2. Let φ be an element of $S(G)$, let us remark that if A_x is a differential operator with polynomial coefficients then the adjoint A_x^* of A_x is also a differential operator with polynomial coefficients, then as in 1., if D is a differential operator with polynomial coefficients we have:

$$\left|(D\psi, \varphi)_{L^2(G)}\right| = |(\psi, D^*\varphi)| = \left|(\Theta(D^*.\varphi), \alpha(\lambda)\Omega_s.\overline{\Omega_m})_{L^2(F)}\right|$$

by the proposition 1

$$\left|(D\psi, \varphi)_{L^2(G)}\right| = \left|(A_{\lambda,p,q}\Theta(\varphi), \alpha(\lambda)\Omega_1.\overline{\Omega_m})_{L^2(F)}\right|$$
$$= \left|(\Theta(\varphi), A^*_{\lambda,p,q}\alpha(\lambda)\Omega_s.\overline{\Omega_m})_{L^2(F)}\right| \leq \|\varphi\|_{L^2(G)}.C_{D^*\psi}$$

thus $D\psi$ is in $L^2(G)$ and finally ψ in $S(G)$. ∎

THEOREM 5. *Let G be a nilpotent connected and simply connected Lie group and let us suppose that the hypothesis (H) is valid, then $\Theta(S(G))$ is dense in $C^\infty(\mathcal{V}, S(p,q))$.*

Proof. Let $E = C_C^\infty(\mathcal{V}) \otimes S(p,q)$, lemma 2 means: $E \subset \Theta(S(G))$; now $C_C^\infty(\mathcal{V})$ is dense in $C^\infty(\mathcal{V})$ and $C^\infty(\mathcal{V}) \otimes S(p,q)$ is dense in its completion:

$$C^\infty(\mathcal{V})\hat{\otimes}S(p,q) = C^\infty(\mathcal{V}, S(p,q)). \qquad \text{(see[7])}$$

Then our theorem is proved. ∎

Of course the hypothesis (H) is valid in the abelien case. In order to build many new examples, we prove finally:

PROPOSITION 2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, we keep our notations, in particular n is $\dim \mathfrak{g}$. Let $\pi : \mathfrak{g}_n^* \to \mathfrak{g}_{n-1}^*$ be the canonical projection and f in $S(G)$, then:*

If $\dim O^\lambda = \dim O^{\pi(\lambda)}$, $\Theta(x_n.f) = -i\alpha\dfrac{\partial}{\partial\lambda_r}\Theta(f)$. $(\alpha \in C^\infty(\mathcal{V}))$

If $\dim O^\lambda = \dim O^{\pi(\lambda)} + 2$, $\Theta(x_n.f) = -i\alpha\dfrac{\partial}{\partial p_r}\Theta(f)$.

Proof. By induction on $n = \dim \mathfrak{g}$, and using the construction of the function $a(p,q,\lambda)$ eorem 3) ∎

SECTION 3.
Examples

In this section we shall give a few examples where our hypothesis (H) is valid.

1. The Heisenberg algebra

Let $\mathbf{g}$ be the Heisenberg algebra,

$$X \in \mathbf{g}, X = x_1 X_1 + x_2 X_2 + x_3 X_3 \quad \text{with} \quad [X_3, X_2] = X_1, \quad X_1 \text{ central},$$

we denote ξ in $\mathbf{g}^*$ as:

$$\xi = \xi_1 f_1 + \xi_2 f_2 + \xi_3 f_3 \quad \text{with} \quad \langle f_i, X_j \rangle = \delta_{i,j},$$

then, in the subset F of $\mathbf{g}^*$ defined by:

$$F = \{\xi \text{ such that } \xi_1 \neq 0\},$$

we have:

$$\tilde{X}_1(p,q,\lambda) = \xi_1 = \lambda, \tilde{X}_2(p,q,\lambda) = \lambda q, \tilde{X}_3(p,q,\lambda) = p$$

thus:

$$\tilde{X}(p,q,\lambda) = x_3 p + \lambda x_2 q + \lambda x_1 = -a(X,p,q,\lambda)$$

then using the formula

$$\Theta(f)(p,q,\lambda) = \int_{\mathbf{g}} e^{ia(X,p,q,\lambda)} f(\exp X) \mathrm{d} X$$

we obtain:

$$\Theta(x_1 f)(p,q,\lambda) = \left(\frac{\partial}{\partial\lambda} + \frac{i}{\lambda} q \cdot \frac{\partial}{\partial q}\right) . \Theta(f)(p,q,\lambda)$$

$$\Theta(x_2 . f)(p,q,\lambda) = \left(-\frac{1}{\lambda}\frac{\partial}{\partial q}\Theta(f)(p,q,\lambda)\right)$$

$$\Theta(x_3 f)(p,q,\lambda) = \frac{\partial}{\partial p}\Theta(f)(p,q,\lambda).$$

2. The $\mathbf{g}_{5,4}$ case

Let $\mathbf{g}$ be the nilpotent Lie algebra generated by $\{X_1, X_2, X_3, X_4, X_5\}$ with:

$$[X_4, X_3] = X_1, [X_5, X_3] = X_2, [X_5, X_4] = -X_3$$

$\mathbf{g}$ is called $\mathbf{g}_{5,4}$ by Dixmier and is a classical counterexample for similar questions [9]. Then in the subset F of $\mathbf{g}^*$ defined by:

$$F = \{\xi \in \mathbf{g}^* \text{ such that } \xi_1 \cdot \xi_2 \neq 0 \ (\xi_i = \langle \xi, X_i \rangle)\},$$

we have:

$$\tilde{X}_1 = \lambda_1, \tilde{X}_2 = \lambda_2, \tilde{X}_3 = \lambda_1 q, \tilde{X}_4 = p, \tilde{X}_5 = \lambda_3 + \frac{\lambda_2}{\lambda_1} p - \frac{1}{2} \lambda_1 q^2.$$

Thus:

$$a(X, p, q, \lambda) = -\tilde{X}(p, q, \lambda) + \frac{\lambda_2}{12} x_4 . x_5 + \frac{\lambda_1}{24} x_5 . x_4^2 + \frac{\lambda_2^2}{24 \lambda_1} x_5^3$$

which proves that hypothesis (H) holds for $\mathbf{g}$ with the same argument as for the Heisenberg algebra.

3. A very special algebras

We shall study now a class of nilpotent algebras called here «very special algebras» for which our hypothesis (H) holds. In fact each nilpotent algebra $\mathbf{g}$ can be embedded in some very special algebra.

CONSTRUCTION. Let n be an integer, for $1 \leq i < j \leq n$ we denote by: $E_{i,j}$ the $n \times n$ matrix with

$$E_{i,j} = (a_{k,\ell}), \qquad a_{k,\ell} = \delta_{i,k} . \delta_{j,\ell} .$$

$\underline{t}(n)$ the space of matrices generated by $\{E_{i,j} | \leq i < i \leq n\}$ and we define a total order on $\mathbb{N} \times \mathbb{N}$ by:,

$$(i, j) \leq (k, \ell) \text{ if } (i \leq k) \text{ or } (i = k \text{ and } j \geq \ell).$$

Let B the vector $(B_1, \ldots, B_n)$ with B in $\mathbb{R}^{\frac{n(n+1)}{2}}, B_k$ in $\mathbb{R}^{n-k+1}$. we put, for $1 \leq i \leq j \leq n$,

$$X_{i,j} = (B_1, \ldots, B_n) \quad \text{where} \quad B_s = 0 \text{ if } s \neq i$$

and $B_i = {}^t(0, \ldots, 0, 1, 0, \ldots, 0)$, the «1» being at the $j - i + 1^{\text{th}}$ place.

If A is in $\mathfrak{t}(n)$, we define by induction the matrices:

$$A_1 = A, A_{i+1} = \begin{bmatrix} 0 & * \dots * \\ 0 & A_i \end{bmatrix}$$

Let (A, B) be the matrix:

$$(A,B) = \begin{bmatrix} A_1 & 0 & \dots & \dots & 0 & B_1 \\ 0 & A_2 & 0 & \dots & 0 & B_2 \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & A_n & B_n \\ 0 & \dots & \dots & \dots & & 0 \end{bmatrix}$$

The space:

$$\mathfrak{g}\{(A,B) s.t A \in \mathfrak{t}(n), B \in \mathbb{R}^{\frac{n(n+1)}{2}}\}$$

is a nilpotent Lie algebra. Its bracket is:

$$[(A,B),(A',B')] = ([A,A'], A_1 B'_1 - A'_1 B_1, \dots, A_n B'_n - A'_n B_n) \quad (3,1)$$

thus:

For $1 \leq i \leq n$ or $1 \leq i < j \leq n$, we put:

$\mathfrak{g}_{i,j}$ is the Lie algebra generated by $(0, X_{k,k})$ for $k \leq i$

$\mathfrak{g}_{i,j}$ is the Lie algebra generated by $(0, X_{k,s})$ with $(k,s) \geq (i,j)$

$\mathfrak{g}'_{i,j}$ is the Lie algebra generated by $(E_{k,s}, 0)$ and $(0, B)$ with B is in $\mathbb{R}^{n(n+1)/2}$ and $(k,s) \leq (i,j)$

then:

$$\{0\} \subset \mathfrak{g}_{1,1} \subset \dots \subset \mathfrak{g}_{n,n} \subset \mathfrak{g}_{n-1,n} \subset \mathfrak{g}_{n-2,n-1} \subset \mathfrak{g}_{n-2,n} \subset \mathfrak{g}_{n-3,n-2} \subset \mathfrak{g}_{n-3,n-1}$$
$$\subset \mathfrak{g}_{n-3,n} \subset \dots \subset \mathfrak{g}_{1,n} \subset \mathfrak{g}'_{1,n} \subset \dots \subset \mathfrak{g}'_{1,2} \subset \mathfrak{g}'_{2,n} \subset \dots \subset g'_{2,3} \subset \dots$$
$$\dots \subset \mathfrak{g}'_{n-1,n} = \mathfrak{g} \qquad (3,2)$$

is a sequence of ideals of $\mathfrak{g}$, we denote it by $\{0\} \subset \mathfrak{g}_1 \subset \dots \subset g_{n^2}$.

Let $F \subset \mathfrak{g}^*$ the set $F = \{\xi \in \mathfrak{g}^* s.t \tilde{X}_i(\xi) \neq 0 \text{ for } i = 1, \dots, n\}$

LEMMA 3. *For our algebra with this sequence of ideals (3, 2), The parametrization of* F *introduced in theorem 1, has the following property: If we put*

$$d_1 = 1,\ d_2 = 2, \ldots, d_n = n, \quad c_k = n+1, c_{k-1} = n+2, \ldots, c_1 = \frac{(n(n+1))}{2}$$

$$\ell_1 = \frac{n(n+1)}{2} + 1, \ell_2 = \frac{n(n+1)}{2} + 2, \ldots, \ell_k = n^2 \text{ and :}$$

$$Z_{d_i} = \dot{\lambda}_i,\ Z_{c_i} = q_i, \quad Z_{\ell_i} = p_i, \text{ then :}$$

For $i = 1, \ldots\ldots, n^2$:

$$\tilde{X}_i(\lambda, p, q) = \tilde{X}_i(Z) = Z_i . f_i(Z_d) + g_i(Z_1, \ldots, Z_{i-1}).$$

where f_i *is a rational, regular on* $\mathcal{V}$ *and* g_i *is a polynomial function of the variable* z_c, z_ℓ *and rational function of the variables* Z_d.

Proof. By induction on $n = \dim \mathbf{g}$, following the sequence (3; 2), and using the parametrisation (theorem 1) ∎

LEMMA 4. *We keep the preceding notations then: for* $m = n^2$

$$a(X, Z) = \sum_{i=1}^{m} x_i Z_i f_i(Z_d) + \sum_{(a_1, \ldots, a_m)} x_1^{a_1} \ldots x_m^{a_m} f_{a_1}, \ldots, a_m(Z)$$

where $f_{a_j}, \ldots, a_m(Z) = g(Z_1, \ldots, Z_{j-1})$ (g *is rational in* Z_d, *polynomial in* Z_c, Z_ℓ)

Proof. Let $\mathbf{g}_0 = \{0\} \subset \mathbf{g}_1 \subset \ldots \subset \mathbf{g}_m = \mathbf{g}$ being the sequence of ideal of $\mathbf{g}$, then:

$$\forall X \in \mathbf{g}, \tilde{X}(Z) = \sum_{i=1}^{m} x_i \tilde{X}_i$$

thus:

$$\tilde{X}(Z) = \sum_{i=1}^{m} x_i Z_i f(Z_d) + \sum_{i=1}^{m} x_i g_i(Z_i, \ldots, Z_{i-1})$$

$$\forall X \in \mathbf{g}, (x_i \in \mathbb{R}) \quad (3,3)$$

That means that what we write in theorem 1:

$\tilde{X}(Z) = \sum_{j=1}^{k} \alpha_j(X, Z) Z_{\ell_j} + \alpha_0(X, Z)$ where $\alpha \in \mathbb{R}(Z_d)[Z_{c_{j+1}}, \ldots, Z_{c_k}]$, and

$$1 = d_1 < \ldots < d_r < c_k < \ldots < c_1 < \ell_1 < \ldots < \ell_k = m = n^2 .$$

Read here

$$\alpha_j(X, Z) = x_{\ell_j}.f_{\ell_j} + \sum_{i>\ell_j} x_i.\mathbf{g}_i(Z_1, \ldots, Z_{c_{j,1}})$$

Let us compute the function a, we obtain:

$$a(X, Z) = \sum_{i=1}^{m} x_i Z_i, f_i(Z_d) + \sum_{(a_1, \ldots, a_m)} x_1^{a^1} \ldots x_m^{a_m} f_{a_1}, \ldots, a_m(Z)$$

where $f_{a_j}, \ldots, a_m(Z) = g(Z_1, \ldots, Z_{j-1})$ (g is rational in Z_d, polynomial in Z_c, Z_ℓ)

■

PROPOSITION 3. *We keep our preceding notations, let $\mathbf{P}(x_1, \ldots, x_m)$ be a polynomial function on G, then there exists a differential operator A_Z with coefficients rational in Z_d and polynomial in (Z_c, Z_ℓ), such that:*

$$\Theta(\mathbf{P}(x).f) = A_Z.\Theta(f)$$

Proof. It's the consequence of lemma 3. ■

4. The 3 step nilpotent Lie algebra (there have been studed by Ratcliff in [8])

Let H_n be the Heisnberg group of dimension $2n+1$, with Lie algebra $\mathbb{H}_n$, and S the sub group of the symplectic group SP_{2n}

$$S = \left\{ \begin{bmatrix} I, & 0 \\ S, & I \end{bmatrix}, I \text{ is the identité and } S \text{ a } n \times n \text{ symmetric matrix} \right\}$$

with Lie algebra S.

The semi-direct product of S and $\mathbb{H}_n$ is a 3 step nilpotent algebra with the following bracket:

For X, Y in $\mathbf{R}^n, Z$ in $\mathbf{R}$ and S a $n \times n$ symetric matrix, we write (S, X, Y, Z) as an element of $S\mathbb{H}_n$ and

$$((S, X, Y, Z), (S', X', Y', Z')) = (0, 0, S.X' - S'.X, X.Y' - X'Y)$$

If we put: $B_i = (0, \ldots, 1, 0, \ldots, 0)$, 1 at the i^{th} place, $S_{i,j} = (a_{k,\ell})$ the symetric matrix with

$$(a_{k,l} = \delta_{i,k} \cdot \delta_{j,l} + \delta_{i,l} \cdot \delta_{j;k}); \; X_i = (0, B_i, 0, 0);$$

$$Y_i = (0, 0, B_i, 0); \; Z_0 = (0, 0, 0, 1)$$

and and $E_{i,j} = (S_{i,j}, 0, 0, 0)$ then the system $\{Z_0; X_i, Y_i, E_{i,j}; 1 \le i \le j \le n\}$ is a basis of $S\mathbb{H}_n$ and: $(E_{i,j}X_i) = Yj; (E_{i,j}, X_j) = Y_i; (X_i, Y_i) = Z_0^-$ We define the total order on $\mathbb{N}^2$ by: $(i,j) \le (r,s)$ if $(i < r)$ or $(i = r$ and $j \le s)$.$\mathbb{H}$ Let for $1 \le i \le j \le n$: – $\mathfrak{g}_0$ be the Lie algebra generated by Z_0

– $\mathfrak{g}_i$ be the Lie algebra generated by $Z_0; Y_1; \ldots; Y_{n-i+1}$

– $\mathfrak{g}_i'$ be the Lie algebra generated by $Z_0, Y_1, \ldots, Y_n, X_1, \ldots, X_i$

– $\mathfrak{g}_{i,j}$ be the Lie algebra generated by $Z_0, Y_1, \ldots, Y_n, X_1, \ldots, X_n, E_{r,s}$ for $(i,j) \le (r,s)$

Then: $\{0\} \subset \mathfrak{g}_0 \subset \mathfrak{g}1 \subset \ldots \subset \mathfrak{g}_n \subset \mathfrak{g}_1' \subset \ldots \subset \mathfrak{g}_n' \subset \mathfrak{g}_{1,1} \subset \ldots \subset \mathfrak{g}_{n,n} = \mathfrak{g}$ is a sequence of ideals of $S\mathbb{H}_n$, then we have:

$$a(X,p,q,\lambda) = \tilde{X}(p,q,\lambda) \quad \frac{1}{24\lambda}\sum_{i=1}^{n} x_{i,i}x_i^2 - \frac{1}{12\lambda_1}\sum_{i<j} x_{i,j}x_ix_j$$

COROLLARY. *For the nilpotent algebra* $S\mathbb{H}_n$, *the hypothesis* (H) *holds.*

Let $\mathfrak{g}$ be any 3- step nilpotent Lie algebra with one dimensional center, in [8] it is proved there exists for some n an injection J from $\mathfrak{g}$ into $S\mathbb{H}_n$ such that $J(\mathfrak{g})$ is an ideal of $S\mathbb{H}_n$, and:

$$J(\mathfrak{g}) + S = S\mathbb{H}_n$$

Now let V be a subspace of S such that $\mathfrak{g} \simeq \mathbb{H}_n + V$ and $\{e_1, \ldots, e_m\}$ a basis of V, we choose a basis $\{e_1, \ldots, e_m'\}$ of S such that $\{e_1, \ldots, e_m\} \subset \{e_1, \ldots e_m'\}$. Thus if we put: – $\mathfrak{g}_0$ the Lie algebra generated by Z_0

– $\mathfrak{g}_i'$ the Lie algebra generated by $Z_0, Y_1, \ldots, Y_n, X_1, \ldots, X_i$

– $\mathfrak{g}_I''$ the Lie algebra generated by $Z_0, Y_1, \ldots, Y_n, X_1, \ldots, X_n, e_1, \ldots, e_i$

and since $[S, S] = 0$ then:

$$\{0\} \subset \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \ldots \subset \mathfrak{g}_n \subset \mathfrak{g}_1' \subset \ldots \subset \mathfrak{g}_n' \subset \mathfrak{g}_1'' \subset \ldots \subset \mathfrak{g}_m'' = \mathfrak{g}$$

is a sequence of ideals of $S\mathbb{H}_n$ and we have a parametrization of $S\mathbb{H}_n(p,q,\lambda)$ with:

$\lambda_s = \sum \beta_{i,j}\lambda_{i,j}$ $\quad (\beta_{i,j}$ is in $\mathbb{R}, \lambda_{i,j}$ is the invariant rational of $S\mathbb{H}_n)$.

Finally, the construction of the function a on $\mathfrak{g}$ using J proves as before:

PROPOSITION 5. *Let* $\mathfrak{g}$ *be a 3 step nilpotent Lie algebra then the hypothesis* (H) *holds for* $\mathfrak{g}$.

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